2022

Time - 3 hours

Full Marks - 80

*Answer **all groups** as per instructions.*
Figures in the right hand margin indicate marks.
The symbols used have their usual meaning.

GROUP - A

1. Answer all questions and fill in the blanks as required. [1 × 12

(a) A metric space (X, d) is said to be _______ if every cauchy sequence in X is convergent.

(b) The intersection of any finite family of open sets is open. (True / False)

(c) Let (X, d) be a metric space and A, B be subsets of X. Then which is true.

(i) $(A \cap B)^0 = A^0 \cap B^0$

(ii) $(A \cup B)^0 \subseteq A^0 \cup B^0$

(iii) $\overline{A \cap B} = \overline{A} \cap \overline{B}$

(iv) None of these

(d) The irrationals in $\mathbb{R}$ are of category ______.

(e) In discrete metric space (X, d), if x, y $\in$ X and x $\neq$ y, then d(x, y) = ______.

(f) If A is a closed subset of metric space (X, d), then A = ______.

(g) A point x $\in$ X is called a fixed point of the mapping $T : X \rightarrow X$ if T_x = ______.

(h) $Q \subset \mathbb{R}$ is a connected set. (True / False)

(i) The function $f : (0, 1) \rightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous but not bounded on (0, 1). (True / False)

(j) C[0,1] stands for what ?

(k) Every compact metric space X is separable. Is it True of False ?

(l) Let f be a continuous function from a compact metric space (X, d_X) into a metric space (Y, d_Y). Then the range f(X) of f is also ______.

GROUP - B

2. Answer <u>any eight</u> questions. [2 × 8

(a) Define open ball and closed ball.

(b) Prove that, let (X, d) be a metric space and A and B be sub-sets of X, then $A \subseteq B \Rightarrow A^0 \subseteq B^0$.

(c) Show that the set Z of integers is a closed subset of the real line.

(d) Prove that the empty set ϕ is both open and closed.

(e) Define continuous function on a metric space.

(f) Define uniform convergence of an sequence.

(g) Define nowhere dense subset.

(h) Prove that, if Y is a connected set in a metric space (X, d) then any set Z such that $Y \subseteq Z \subseteq \overline{Y}$ is connected.

(i) Define compact metric space.

(j) State contraction Mapping Principle.

GROUP - C

3. Answer any eight questions. [3 × 8

(a) Let (X, d) be a metric space. Show that any intersection of closed sets is closed.

(b) Prove that in any metric sapce (X, d), each open ball is an open set.

(c) Let (X, d) be a metric space and Y a subspace of X. Let $z \in Y$ and $r > 0$. Then prove that $S_Y(z, r) = S_X(z, r) \cap Y$, where $S_Y(z, r)$ and $S_X(z, r)$ denote the ball with centre z and radius r in Y and in X respectively.

(d) Let Y be a subspace of a metric space (X, d). Then prove that every subset of Y that is open in Y is also open in X if and only if Y is open in X.

(e) Prove that in any metric space, there is a countable base at each point.

(f) Let $f : \mathbb{R} \to \mathbb{C}$ be defined by $f(x) = x + ix^2$. Verify that f is continuous at x = 2.

(g) Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces and let $f : X \to Y$ and $g : Y \to Z$ be continuous. Prove that the composition map gof is a continuous map from X to Z.

(h) Prove that A mapping $f : X \to Y$ is continuous on X, then $f^{-1}(F)$ is closed in X, for all closed subsets F of Y.

(i) Let $K \subseteq \mathbb{R}$ both closed and bounded and M = sup K, m = inf K. Then prove that M and m are in K.

(j) Let (X, d) be a metric space and (X, d) is disconnected then prove that there exist two nonempty disjoint subsets A and B both open in X, such that $X = A \cup B$.

GROUP - D

4. Answer any four. [7 × 4

(a) State and prove Cantor's theorem.

(b) Show that, let (X, d) be metric space and $F \subseteq X$. Then the following statements are equivalent.

(i) $x \in \bar{F}$

(ii) $S(x, \varepsilon) \cap F \neq \phi$ for every open ball $S(x, \varepsilon)$ centred at x.

(iii) $\exists$, an infinite sequence $\{x_n\}$ of points of F such that $x_n \to x$

(c) State and prove Baire's category theorem.

(d) Show that, if (X, d) be a metric space. The following statements are equivalent.

(i) (X, d) is separable.

(ii) (X, d) satisfies the second axiom of countability.

(iii) (X, d) is Lindelof.

(e) A mapping $f : X \to Y$ is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subsets G of Y.

(f) Let $\{f_n\}_{n \geq 1}$ a sequence of functions defined on a metric space (X, d_X) with values in a complete space (Y, d_Y). Then prove that there exists a function $f : X \to Y$ such that $f_n \to f$ uniformly on X. If and only if following condition is satisfied for every $\varepsilon > 0$, there exists an integer n_0 such that $m, n \geq n_0$ implies $d_Y(f_m(x), f_n(x)) < \varepsilon$ for every $x \in X$.

(g) Let (X, d) be a metric space and Y a subset of X. If Y is a compact subset of X, then prove that Y is closed and bounded.