No. of Printed Pages : 4 Sem-IV-Math-CC-X(R&B)

2022

Time - 3 hours

Full Marks - 80

*Answer **all groups** as per instructions.*
Figures in the right hand margin indicate marks.
The symbols used have their usual meaning.

GROUP - A

1. Answer all questions and fill in the blanks as required. [1 × 12

(a) A commutative ring R is called a _____ if $a^2 = a$ for all $a \in R$.

(b) A finite integral domain is a field. (True / False)

(c) _______ is the only nilpotent element in an integral domain.

(d) The inverse image of a prime ideal is maximal. (True / False)

(e) Is the ring 2Z isomorphic to the ring 4Z ?

(f) Define ring homomorphism.

(g) Are there any nonconstant polynomials in Z[x], that have multiplicative inverse ? (Yes / No)

(h) Find the sum of the polynomial ring if

$f(x) = x + 1, \; g(x) = x + 1$, in $Z_2[x]$.

(i) The product of two primitive polynomials is ________.

(j) Every principal ideal domain is a unique factorization domain. (True / False)

(k) Is a sub-ring of a UFD a VFD ? (Yes / No)

(l) If D is a UFD, then D(F) is a UFD. (True / False)

GROUP - B

2. Answer any eight questions. [2 × 8

(a) If R be a ring, then show that the zero element is unique.

(b) Find all idempotent elements in $Z_3 \oplus Z_6$.

(c) Show that $2Z \cup 3Z$ is not a subring of Z.

(d) If R is a ring with unity and N is an ideal of R containing a unit, then show that N = R.

(e) Find all maximal ideals of $Z_8 \oplus Z_{30}$.

(f) Find multiplicative inverse of the polynomial $2x + 1$ in $Z_4[x]$.

(g) If $f(x) = x^3 + 2x^2 - x + 1$ and $g(x) = x + 1$ in $Z_3[x]$, determine the quotient and remainder upon dividing f(x) by g(x).

(h) If D be an integral domain and f(x), g(x) $\in$ D[x], then show that $\deg(f(x) \cdot g(x)) = \deg f(x) + \text{dex}\, g(x)$.

(i) Show that every field is a Euclidean domain.

(j) Show that the polynomial $x^2 + 1$ is irreducible over Z_3 but reducible over Z_5.

GROUP - C

3. Answer any eight questions. [3 × 8

(a) Determine $U(Z[x])$.

(b) Prove that in the ring Z_n, the divisors of O are precisely those non zero elements that are not relatively prime to n.

(c) Prove that the characteristic of an integral domain is either zero or prime.

(d) Show that $\phi : C[0, 1] \to R \oplus R$ defined by $\phi(f) = (f(0), f(1))$ is a homomorphism.

(e) Prove that the sum of the squares of three consecutive integers cannot be a square.

(f) Show that Q is not isomorphic to $Q[\sqrt{2}]$.

(g) Find all irreducible polynomials of degree 3 in $Z_2[x]$.

(h) Construct field of order 27.

(i) Show that Z[x] is not a PID. (Principal Ideal Domain).

(j) Give an example of a unique factorization domain with a subdomain that does not have a unique factorization.

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GROUP - D

*Answer **any four** questions.*

4. State and prove Existence of factor rings. [7

5. Show that the ring $Z[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in Z\}$ is an integral domain. [7

6. State and prove first isomorphism theorem for Rings. [7

7. Prove that if R be a commutative ring with unity and let A be an ideal of R. Then R/A is a field iff A is maximal. [7

8. State and prove Division algorithm for F[x]. [7

9. Show that $Z_3[x]/\langle x^2+1\rangle$ is isomorphic to $Z_3[i] = \{a_i + b \mid a, b \in Z_3\}$. [7

10. Prove that in a principal ideal domain, an element is prime if and only if it is irreducible. [7

11. Show that the ring of Gaussian integers $Z[i] = \{a + b_i \mid a, b \in Z\}$ is an Euclidean domain. [7