No. of Printed Pages : 5

Sem-V-Math-CC-12
(Reg&Back)

2023-24

Time - 3 hours

Full Marks - 80

*Answer **all groups** as per instructions.*
Part of each question should be answered continuously.
Figures in the right hand margin indicate marks.
Candidates are required to answer
in their own words as far as practicable.
The symbols used have their usual meanings.

GROUP – A

1. Answer all questions and fill in the blanks as required. [1 × 12

(a) In a vector space V, $ax = ay \Rightarrow x = y$.
(Write true or false.)

(b) {0} is linearly dependent. Prove it.

(c) Span (ϕ) = ________.

(d) dim $P_n(F)$ = ________.

(e) $P_m(F) \approx P_n(F) \Leftrightarrow m = n$.
(Write true or false.)

(f) Standard basis for R^3 is ________ .

(g) $P_3(R) \approx M_{2\times 2}(R)$. (Write true or false.)

(h) Every orthonormal set is ________ .

(i) The orthogonal matrix of $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is ________ .

(j) Every ________ operator is diagonalizable.

(k) $(R(T))^{\perp} =$ ________ .

(l) State Rank-nullity theorem.

GROUP – B

2. Answer <u>any eight</u> of the following questions. [2 × 8

(a) If V is a vector space over F, then show that

$$(-a)x = -(ax),\ \forall x \in V \text{ and } a \in F.$$

(b) If $u = (1, -2, k) \in R^3$ be a linear combination of the vectors $(3, 0, -2), (2, -1, -3)$; then find k.

(c) Show that the set of vectors $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ generates R^3.

(d) If $v = (1 - i, 2 + 3i)$, then find $\|v\|$.

(e) Define T-covariant.

(f) Let $T : P_3(R) \to P_2(R)$ be defined by $T(f(x)) = f'(x)$. Then find $[T]_{\beta}^{\gamma}$, where β, γ are the standard bases.

(g) Define orthonormal basis.

(h) Prove that $(T^*)^* = T$.

(i) Define orthogonal projections on W.

(j) If $T : V \to V$ be a linear transformation on a vector space V, then prove that N(T) is T-invariant.

GROUP – C

3. Answer <u>any eight</u> questions. [3 × 8

(a) Show that

$$B = \{(1, 1), (-1, 1)\} \text{ is a basis for } R^2.$$

(b) Give an example of distinct linear transformations T and U such that $N(T) = N(U)$ and $R(T) = R(U)$.

(c) Let $T : V \to W$ be a linear map. Then prove that

$$T \text{ is one-one} \Leftrightarrow N(T) = \{0\}.$$

(d) If $T : V \to W$ is linear and invertible, then prove that

$$T^{-1} : W \to U \text{ is linear and invertible.}$$

(e) Show that if A and B are simultaneously diagonalizable metrices, then A and B commute.

(f) Let V be an inner product space.

Then prove that $\left| \|x\| - \|y\| \right| \leq \|x - y\|$ for x, y $\in$ V.

(g) Show that $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

(h) Find an orthogonal matrix whose 1st row is $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$.

(i) If A, B are similar n × n matrices,

then prove that $l_r(A) = l_r(B)$.

(j) Determine that the matrix $\begin{bmatrix} 1 & i \\ 1 & 2+i \end{bmatrix}$ is normal.

GROUP – D

4. Answer any four questions. [7 × 4

(a) If V be a vector space over F, then prove that arbitrary intersection of subspaces of V is a subspace of V.

(b) If V be a finite dimensional vector space and U, W are two subspaces of V ; then prove that

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

(c) Let T and U be self-adjoint operators on an inner product space V. Prove that

TU is self-adjoint $\Leftrightarrow$ TU = UT.

(d) If $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 0 \end{pmatrix}$ and $\gamma = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is an ordered basis for R^3, then find $(L_A)_\gamma$

(e) Find the minimal solution of the system of linear equations

$$x + y - z = 0$$

$$2x - y + z = 3$$

$$x - y + z = 2.$$

(f) State and prove Cayley-Hamilton theorem.

(g) Let $T : R^3 \to R^2$ be defined by $T(x, y, z) = (3x + 2y - 4z, x - 5y + 3z)$. Find the matrix T relative to the bases of R^3 and R^2 given by

$\beta_1 = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ and $\beta_2 = \{(1, 3), (2, 5)\}$.